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Logarithmic Asymptotics for the *GI/G/1*-type Markov Chains and their Applications to the *BMAP/G/1* Queue with Vacations

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Abstract—We study tail asymptotics of the stationary distribution for the $GI/G/1$-type Markov chain with finitely many background states. Decay rate in the logarithmic sense is identified under a number of conditions on the transition probabilities. The results are applied to the $BMAP/G/1$ queue with vacations. The relationship between vacation time and the decay rate of the queue length distribution is investigated.

Keywords The *GI/G/*1-type Markov chain, Stationary distribution, Tail asymptotics, Logarithmic asymptotics, Matrix-analytic methods, Vacation queue, *BMAP*

1. INTRODUCTION

Queueing models have been used in the design of telecommunications, manufacturing, service, and healthcare systems. To analyze a queueing model, Markov chains are usually utilized. While explicitly or numerically tractable solutions have been obtained for a number of Markov chains with a special structure (e.g., Cohen, 1982; Neuts, 1981, 1989; and Tian and Zhang, 2006), it is more challenging to solve general Markov chains. For many design and control problems, on the other hand, approximation results are good enough for applications. For example, the loss probability, which plays an important role in the design of queueing systems, can be approximated by tail asymptotics. In recent years, the study of tail asymptotics of Markov chains has attracted the attention of researchers and practitioners (e.g., Whitt, 1993; Foley and McDonald, 2005; Kimura et al., 2012, 2013; Li and Zhao, 2005a,b; Masuyama, 2011; Miyazawa, 2009; and references therein.) In this paper, we study the tail asymptotics of the $GI/G/1$ type Markov chains, and apply the results to analyze vacation queues.

We consider a two-dimensional Markov chain $\{(X_n, Y_n)\colon n = 0, 1, 2, \ldots\}$ with finitely many background states defined on state space $S = \{(0, i), i = 1, \ldots, m_0\} \bigcup \{(n, j), n = 1, 2, \ldots, j = 1, \ldots, m\}$, where X_n is the level variable, and Y_n is the background (phase) variable. The set of states $\{(0, i), i = 1, \ldots, m_0\}$ is called level zero, and the set of states $\{(n, i), i = 1, \ldots, m\}$ is called level n, for $n \geq 1$. We assume that the Markov chain is homogeneous along the level direction except for level zero. By the partition according to levels, the transition probability matrix P of the Markov chain $\{(X_n, Y_n)\colon n = 0, 1, 2, \ldots\}$ can be expressed in block matrix form:

$$P = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ B_{-1} & A_0 & A_1 & A_2 & \cdots \\ B_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ B_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad (1)$$

where B_0 is an $m_0 \times m_0$ matrix, $\{B_n, n = 1, 2, \ldots\}$ are $m_0 \times m$ matrices, $\{B_{-n}, n = 1, 2, \ldots\}$ are $m \times m_0$ matrices, and $\{A_n, n = 0, \pm 1, \pm 2, \ldots\}$ are $m \times m$ matrices. For convenience, we assume that $m_0 = m$. Each block is a collection of transition probabilities for transitions from some level to another. We call this type of Markov chains the $GI/G/1$-type Markov chain. The $GI/M/1$-type Markov chain, the $M/G/1$-type Markov chain and the quasi birth-and-death process (QBD) are three special examples whose transition matrices satisfy the properties, respectively: (1) both

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$A_n = 0$ and $B_n = 0$ for $n \geq 2$, (2) both $A_n = 0$ and $B_n = 0$ for $n \leq -2$, and (3) both $A_n = 0$ and $B_n = 0$ for $n \geq 2$ and $n \leq -2$. The state space of the Markov chain P, based on the above partition of states, can be expressed as $S = \bigcup_{n=0}^{\infty} L_n$ with $L_n = \{(n,1),(n,2),\cdots,(n,m)\}$, for $n = 0, 1, \ldots$. Define $L_{\leq n} = \bigcup_{l=0}^{n} L_l$, and $L_{\geq n} = \bigcup_{l=n}^{\infty} L_l$. Define the following generating functions: for $z \in C$ (the set of all complex numbers),

$$A^*(z) = \sum_{n=-\infty}^{\infty} z^n A_n, \quad A_+^*(z) = \sum_{n=1}^{\infty} z^n A_n \text{ and } B_+^*(z) = \sum_{n=1}^{\infty} z^n B_n. \tag{2}$$

Let

$$\phi_{A_+} = \min_{1\leq i,j\leq m} \sup\{|z| \geq 1: |[A_+^*(z)]_{i,j}| < \infty\},$$
$$\phi_{B_+} = \min_{1\leq i,j\leq m} \sup\{|z| \geq 1: |[B_+^*(z)]_{i,j}| < \infty\}. \tag{3}$$

Note that we shall use $[X]_{i,j}$ to denote the (i,j)th entry of a matrix X throughout the paper.

Define $\chi(z)$ as follows: (a) for $0 \leq |z| < \phi_{A_+}$, $\chi(z)$ is the eigenvalue of $A^*(z)$ with the greatest real part; (b) at $|z_0| = \phi_{A_+}$, $\chi(z_0) = \lim_{z\to z_0:\ |z|<|z_0|} \chi(z)$. In this paper, we fix $\alpha > 1$ (if it exists) such that there exists $|z| = \alpha$ satisfying that $\chi(z) = 1$ and $A^*(z)$ is finite. Whenever α is used, we technically assume that α exists.

Throughout this paper, the $GI/G/1$-type Markov chain is assumed to be aperiodic, irreducible and positive recurrent. Under this assumption, its stationary distribution $\boldsymbol{\pi}$ exists uniquely and is positive element-wise. Partition $\boldsymbol{\pi}$ into $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots)$ according to the levels, where $\boldsymbol{\pi}_n = (\pi_{n,1}, \pi_{n,2}, \ldots, \pi_{n,m}), n = 0, 1, \ldots$. This paper is concerned with asymptotics of $\boldsymbol{\pi}_n$, when n is large. More specificaly, we investigate the limit of $\ln \pi_{n,k}$ when $n \to \infty$, which is called the logarithmic asymptotics.

The study of logarithmic asymptotic, in general, is of great interest because of not only theoretical challenges but also its potential applications, for instance, in the design and control of high-speed communication networks. The investigations of logarithmic asymptotic of the stationary distribution of a Markov chain include exact geometric decay (a special case of logarithmic asymptotic that implies logarithmic asymptotic (not vice versa)), logarithmic asymptotic in the general sense, and relevant applications. Many studies were contributed for exact geometric asymptotic of the stationary distribution of $M/G/1$-type Markov chain with finitely many background states. For example, Abate et al. (1994), Falkenberg (1994), and Møller (2001) derived the exact geometric decay rate with progressively refined conditions when $\alpha < \min\{\phi_{A_+}, \phi_{B_+}\}$. Takine (2004) considered periodicity of the level variable and analyzed exact geometric decay when $\alpha < \phi_{B_+}$ and the period is d. Kimura et al. (2010) analyzed exact geometric decay if (1) $\alpha < \min\{\phi_{A_+}, \phi_{B_+}\}$ and the period of the level variable is d, (2) $\phi_{B_+} < \alpha$ and $B_+^*(z)$ is a meromorphic function, or (3) $\phi_{B_+} = \alpha < \phi_{A_+}$ and $B_+^*(z)$ is a meromorphic function. For further extension to the $GI/G/1$-type Markov chain with finitely many background states, Li and Zhao (2005b) analyzed exact geometric decay when $\alpha < \min\{\phi_{A_+}, \phi_{B_+}\}$ and $\{A_n, n = 0, 1, 2, \ldots\}$ is 1-arithmetic. Tai and Zhao (2010) showed that $\{\boldsymbol{\pi}_n, n = 0, 1, 2, \ldots\}$ has an exact geometric decay if (1) $\alpha < \phi_{A_+}$, $\alpha \leq \phi_{B_+}$ and $B_+^*(\alpha) < \infty$; (2) $\alpha = \phi_{A_+}$, and $\alpha < \phi_{B_+}$; (3) $\phi_{B_+} < \alpha$ and $\lim_{n\to\infty} \phi_{B_+}^n B_n = D \gneqq 0$; or (4) $\phi_{B_+} \leq \phi_{A_+}$ and $\lim_{n\to\infty} \phi_{B_+}^n B_n = D \gneqq 0$ (when $A^*(\phi_{A_+}) < \infty, \chi(z) < 1$ for all $1 < |z| \leq \phi_{A_+}$). For the $GI/G/1$-type Markov chain with finitely many background states, the tail asymptotics have not been found for other cases, which is the main issue of interest in this paper.

Contributions on general logarithmic asymptotic include Glynn and Whitt (1994) that analyzed logarithmic asymptotic for the steady state of general waiting time process under a Gartner-Ellis condition of the partial sums of the increment sequence associated waiting time sequence if the increment sequence is strictly stationary. The results were applied to the queue length process of the $GI/G/1$ queue. Nakagawa (2004) gave a sufficient condition for the logarithmic asymptotic of the tail of a complex sequence, based on a singularity analysis of the generating function associated with a complex sequence. Foley and McDonald (2005) and Miyazawa (2009) analyzed the logarithmic asymptotic for special QBD processes with infinitely many background states. Papers on applications of logarithmic asymptotic rate to create effective bandwidths for admission control and other network resource allocation include: Chang et al. (1992), Chang (1993), and Whitt (1993).

In this paper, we characterize logarithmic asymptotic, by weakening conditions for the exact geometric decay cases, of the stationary distribution $\{\boldsymbol{\pi}_n, n = 0, 1, 2, \ldots\}$ for the $GI/G/1$-type Markov chain under some light-tailed assumptions on transition probabilities. It is shown that the logarithmic asymptotic rate is determined by three factors: α, ϕ_{A_+} and ϕ_{B_+}. Our study offers an comprehensive understanding for logarithmic decay for the $GI/G/1$-type Markov chain, and also provides a theoretical basis for applications.

Queueing models with vacations have many applications and have been investigated extensively. For a queueing system with server vacations, apparently, the queue length has much to do with the vacation time. Existing results indicate that the relationship between the vacation time and the queue length is complicated (e.g., Doshi, 1986; Lucantoni et al., 1990; and Tian and Zhang, 2006). Lucantoni et al. (1990) and Lucantoni (1991) provided a comprehensive study on the $BMAP/G/1$ queue with or without vacations. In this paper, we exemplify the theoretical results on the decay of the stationary distributin of the $GI/G/1$-type Markov chain by applying them to the $BMAP/G/1$ queue with vacations. We focus on the relationship between the vacation time and the decay rate of the queue length distribution. We provide a series of examples to explain the impact of vacation time on the asymptotic rate. The results are simple and capture the behavior of the tail of the queue length distribution. It is interesting to observe that decay rate, which is

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given by $(\min(\alpha, \phi_{A_+}, \phi_{B_+}))^{-1}$, if it exists, remains constant if the vacation time is short. The decay rate then changes along with the convergence norm associated with the vacation time.

The rest of this paper is organized as follows. In Section 2, we provide some preliminaries. In Section 3, we analyze the decay rate in the logarithmic sense of the stationary distribution for the $GI/G/1$-type Markov chain. In Section 4, we analyze tail asymptotics of the $BMAP/G/1$ queue with vacations.

2. PRELIMINARIES

In this section, we give a definition of tail asymptotics for the stationary distribution, provide matrix notations and factorization results, and collect some lemmas.

Consider a sequence $\{M_n, n = 1, 2, \dots\}$ of non-negative $m \times m$ matrices satisfying $\sum_{n=1}^{\infty} M_n < \infty$. The sequence $\{M_n, n = 1, 2, \dots\}$ is called light-tailed if, for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$,

$$\sum_{n=1}^{\infty} [M_n]_{i,j} \exp\{\varepsilon n\} < \infty, \quad \text{forsome } \varepsilon > 0,$$

where ε is independent of i and j. We call $\{M_n, n = 1, 2, \dots\}$ heavy-tailed if it is not light-tailed.

Along the same line as in Zhao et al. (1998), we define the R-measures and G-measures for the $GI/G/1$-type Markov chain, which are matrices $R_{l,n}$ for $l < n$ and $G_{l,n}$ for $l > n$, respectively, and will be used to analyze tail asymptotics. $R_{l,n}$ is an $m \times m$ matrix whose (i, j)th entry is the expected number of visits to state (n, j) before hitting any state in $L_{\le(n-1)}$, given that the process starts in state (l, i). $G_{l,n}$ is an $m \times m$ matrix whose (i, j)th entry is the probability of hitting state (n, j) when the process enters $L_{\le(l-1)}$ for the first time, given that the process starts in state (l, i). We call matrices $R_{l,n}$ and $G_{l,n}$ the matrices of the expected numbers of visits to higher levels before returning to lower levels and the matrices of the first passage probabilities to lower levels, respectively. We can write $R_{n-l} = R_{l,n}$ and $G_{n-l} = G_{n,l}$ for $l > 0$ and $n > l$, due to the repeating structure in the matrix P. We define a matrix sequence $\{\Phi_n, n = 0, \pm 1, \pm 2, \dots\}$ as follows. For $n \ge 1$, partition the transition matrix as

$$P = \begin{matrix} & \begin{matrix} L_{\le n} & L_{\ge(n+1)} \end{matrix} \\ \begin{matrix} L_{\le n} \\ L_{\ge(n+1)} \end{matrix} & \begin{pmatrix} Q_0 & U \\ V & Q_1 \end{pmatrix} \end{matrix}.$$

Set $P^{[n]} = (P^{[n]}_{l,k})_{l,k=0,\dots,n} = Q_0 + U\hat{Q}_1 V$, where $\hat{Q}_1 = ([\hat{Q}_1]_{l,k})_{l,k=1,2,\dots} = \sum_{u=0}^{\infty} Q_1^u$. It is shown in Grassmann and Heyman (1990) that the matrices $P^{[n]}_{n-l,n}$, for $0 \le l \le n-1$, and $P^{[n]}_{n,n-k}$, for $0 \le k \le n-1$, are both independent of n, if $n \ge 1$. Hence, for $n \ge 1$, $0 \le l \le n-1$, and $0 \le k \le n-1$, we can define

$$\Phi_l = P^{[n]}_{n-l,n}, \qquad \Phi_{-k} = P^{[n]}_{n,n-k}.$$

The following Winner-Hopf factorization is useful (see Li and Zhao (2005b) for example):

$$I - A^*(z) = (I - R^*(z))(I - \Phi_0)(I - G^*(z)),$$

for z such that $A^*(z)$ is finite, where $R^*(z) = \sum_{n=1}^{\infty} z^n R_n$ and $G^*(z) = \sum_{n=1}^{\infty} z^{-n} G_n$.

Define the generating function for the stationary distribution $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ and the matrix sequence $\{R_{0,n}, n = 1, 2, \dots\}$ as $\boldsymbol{\pi}^*(z) = \sum_{n=1}^{\infty} z^n \boldsymbol{\pi}_n$ and $R_0^*(z) = \sum_{n=1}^{\infty} z^n R_{0,n}$, respectively. Then the stationary distribution $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ can be expressed in terms of the R-measures (e.g., Grassmann and Heyman, 1990):

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 R_{0,n} + \sum_{l=1}^{n-1} \boldsymbol{\pi}_l R_{n-l}, \qquad n \ge 1, \tag{4}$$

and we have

$$\boldsymbol{\pi}^*(z)(I - R^*(z)) = \boldsymbol{\pi}_0 R_0^*(z). \tag{5}$$

This relation is useful for analyzing tail properties of the stationary distribution.

Define $\phi_\pi = \min_{1 \le j \le m} \sup\{|z| \ge 0 : |[\boldsymbol{\pi}^*(z)]_j| < \infty\}$. Let ϕ_R and ϕ_{R_0} be the convergent radii of $R^*(z)$ and $R_0^*(z)$ respectively. Define $r(z)$ as follows: (a) for $0 \le |z| < \phi_R$, $r(z)$ is the eigenvalue with the largest modulus of $R^*(z)$; (b) at $|z_0| = \phi_R$, $r(z_0) = \lim_{z \to z_0 : |z| < |z_0|} r(z)$.

We provide several lemmas for reader's convenience.

Lemma 2.1. *(Theorem 1 in Li and Zhao 2005b) The radii of convergence satisfy $\phi_{A_+} = \phi_R$ and $\phi_{B_+} = \phi_{R_0}$.*

Lemma 2.2. *(Lemma A.4 in Seneta 1981) Let $\{u_i, i = 0, 1, 2, \dots\}$ be non-negative numbers such that, for all $i, j \ge 0$*

$$u_{i+j} \ge u_i u_j.$$

Suppose the set of those integers $i \ge 1$ for which $u_i > 0$ is non-empty and has g.c.d., say d, which satisfies $d = 1$. Then

$$u = \lim_{n \to \infty} (u_n)^{\frac{1}{n}}$$

exists and satisfies $0 < u \le \infty$; further, for all $i \ge 0$, $u_i \le u^i$.

Lemma 2.3. *(Theorem 2.1(i) in Tai and Zhao 2010) Consider an irreducible positive recurrent $GI/G/1$-type Markov chain. Assume that A is irreducible. Then $\limsup_{n\to\infty} \sqrt[n]{\pi_{n,j}} = \limsup_{n\to\infty} \sqrt[n]{\pi_{n,j'}}$ and $\liminf_{n\to\infty} \sqrt[n]{\pi_{n,j}} = \liminf_{n\to\infty} \sqrt[n]{\pi_{n,j'}}$ for any background states j and j'.*

Lemma 2.4. *(Cauchy-Hadamard Theorem in Markushevich 1965) Consider a power series $f(z)$ of a complex sequence $\{c_n\}$ given by $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Denote by r the radius of convergence of the power series. Then, we have*

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$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \frac{1}{r}.$$

Remark 2.1. *It can be shown that if $r > 1$ in Lemma 2.4, then* $\limsup_{n\to\infty}(\sum_{l=n}^{\infty}|c_l|)^{\frac{1}{n}} = \frac{1}{r}$.

3. THE DECAY RATE IN THE LOGARITHMIC SENSE

In this section, we characterize the decay rate in the logarithmic sense of the stationary distribution $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ for the $GI/G/1$-type Markov chain with finitely many background states whose transition probability matrix P is given by (1). We first define the decay rate in the logarithmic sense. For each $k \in \{1, 2, \cdots, m\}$, the upper decay rate in the logarithmic sense $\bar{\Lambda}_k$ of $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ along the level direction and the lower decay rate in the logarithmic sense $\underline{\Lambda}_k$ of $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ along the level direction can be defined by

$$\bar{\Lambda}_k = \limsup_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,k}$$

and

$$\underline{\Lambda}_k = \liminf_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,k},$$

respectively, where $\bar{\pi}_{n,k} = \sum_{l=n}^{\infty} \pi_{l,k}$ and $\pi_{n,k}$ is the kth element of the vector $\boldsymbol{\pi}_n$. If $\bar{\Lambda}_k = \underline{\Lambda}_k \triangleq \Lambda_k$, then Λ_k is referred to as the decay rate in the logarithmic sense of $\{\boldsymbol{\pi}_{n,k}, n = 0, 1, \dots\}$ along the level direction. In particular, if Λ_k is independent of k, we denote it by Λ, which is the decay rate in the logarithmic sense of $\{\boldsymbol{\pi}_n, n = 0, 1, \dots 0\}$ along the level direction. In our study, we assume that $\{A_n, n = 0, 1, \dots\}$ and $\{B_n, n = 0, 1, \dots\}$ are light-tailed, which is equivalent to that $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ is light-tailed (Li and Zhao, 2005b). We also assume that A is irreducible.

To consider the tail asymptotics, we need to study the non-boundary and boundary transition probabilities. More specifically, we consider several cases according to the relationship between α, ϕ_{A_+} and ϕ_{B_+}, that is, (i) $\alpha \le \phi_{B_+}$; (ii) $\phi_{A_+} \le \phi_{B_+}$; and (iii) $\phi_{B_+} < \alpha$. To describe our study well, we collect previous results on exact geometric decay here. If

(1) $\alpha < \phi_{A_+}$, $\alpha \le \phi_{B_+}$ and $B_+^*(\alpha) < \infty$;

(2) $\alpha = \phi_{A_+}$, and $\alpha < \phi_{B_+}$;

(3) $\phi_{B_+} < \alpha$ and $\lim_{n\to\infty}\phi_{B_+}^n B_n = D \gneqq 0$; or

(4) $\phi_{B_+} \le \phi_{A_+}$ and $\lim_{n\to\infty}\phi_{B_+}^n B_n = D \gneqq 0$ (when $A^*(\phi_{A_+}) < \infty$, $\chi(z) < 1$ for $1 < |z| \le \phi_{A_+}$),

then $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ was shown to have an exact geometric decay in Li and Zhao (2005b) and Tai and Zhao (2010), which implies logarithmic decay. In this section, we weaken conditions on exact geometric decay, under which the logarithmic decay exists, but the exact geometric decay may not. Abate et al. (1995) provided an example for which the exact decay does not exist, and logarithmic asymptotics exist.

In the rest of this section, we consider several cases for which the decay rate in the logarithmic sense exists and is identified.

Theorem 3.1. *Assume that there exists $|z| = \alpha > 1$ such that $\chi(z) = 1$ and $A^*(z)$ is finite. If $\alpha \le \phi_{B_+}$, $\alpha < \phi_{A_+}$, $\{A_n, n = 0, \pm 1, \pm 2, \dots\}$ is 1-arithmatic (see Alsmeyer, 1994), and $B_+^*(z)$ is a meromorphic function (see Rudin, 1974) on $|z| \le \alpha + \delta$, for some $\delta > 0$, then the decay rate in the logarithmic sense of $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ along the level direction exists, and is given by*

$$\Lambda = -\ln \alpha.$$

Proof: It was shown in Li and Zhao (2005b) and Tai and Zhao (2010) that $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ has exact geometric decay when $\alpha < \phi_{A_+}$, $\alpha \le \phi_{B_+}$ and $B_+^*(\alpha) < \infty$, which implies that the decay rate in the logarithmic sense exists. Hence, we only need to prove that our theorem is true, when $\alpha < \phi_{A_+}$, $\alpha = \phi_{B_+}$ and $B_+^*(\alpha) = \infty$.

Firstly, we prove that $-\ln \alpha$ is an upper bound on the decay rate of $\{\boldsymbol{\pi}_n, n = 0, 1, 2, \dots\}$ in the logarithmic sense. To do this, we need to analyze the radius of convergence of $\boldsymbol{\pi}^*(z)$. It is easy to see that for $|z| \le 1$, $\boldsymbol{\pi}^*(z)$ is finite. Since we restrict our discussion to the case: $\alpha < \phi_{A_+}$, and $\alpha = \phi_{B_+}$, we have, for $1 < |z| < \alpha$, the inverse of $I - R^*(z)$ always exists and $R_0^*(z)$ is finite on $|z| < \alpha$. Noting that for $1 < |z| < \alpha$, we have $\boldsymbol{\pi}^*(z) = \boldsymbol{\pi}_0 R_0^*(z)[I - R^*(z)]^{-1}$, and $\boldsymbol{\pi}^*(z)$ is finite. On the other hand, we also have $\boldsymbol{\pi}^*(\alpha) = \infty$ elementwise since there is at least one positive entry in each column of $R_0^*(z)$, which may be infinite when $z = \alpha$, and $\det(I - R^*(\alpha)) = 0$. Consequently, α is the radius of convergence of $\boldsymbol{\pi}^*(z)$. Hence, by Lemma 2.4 (Cauchy-Hadamard theorem), we obtain that for each k,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,k} = -\ln \alpha. \tag{6}$$

Next, we show that $-\ln \alpha$ is a lower bound on the lower decay rate of $\{\boldsymbol{\pi}_n, n = 0, 1, 2, \dots\}$ in the logarithmic sense. From the assumption, $B_+^*(z)$ is a meromorphic function on $|z| \le \alpha$. It is easy to know that α is the radius of convergence of $B_+^*(z)$ when $\alpha < \phi_{A_+}$, $\alpha = \phi_{B_+}$ and $B_+^*(\alpha) = \infty$. Hence, from Theorem 2 in Nakagawa (2004), we obtain that for some i_0 and k_0,

$$\lim_{n\to\infty} \frac{1}{n} \ln [\bar{B}_n]_{i_0,k_0} = -\ln \alpha. \tag{7}$$

where $\bar{B}_n = \sum_{l=n}^{\infty} B_l$. From equation (4) and Theorem 12 in Zhao (2000), we have that

$$\boldsymbol{\pi}_n \ge \boldsymbol{\pi}_0 R_{0,n} \ge \boldsymbol{\pi}_0 B_n.$$

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Then,

$$\bar{\boldsymbol{\pi}}_{n,k_0} \geq \boldsymbol{\pi}_{0,i_0}[\bar{B}_n]_{i_0,k_0},$$

which leads to

$$\frac{1}{n}\ln \bar{\pi}_{n,k_0} \geq \frac{1}{n}\ln \pi_{0,i_0} + \frac{1}{n}\ln [\bar{B}_n]_{i_0,k_0}.$$

From equation (7), we have that

$$\liminf_{n\to\infty} \frac{1}{n}\ln \bar{\pi}_{n,k_0} \geq -\ln \alpha.$$

By Lemma 2.3, for each k,

$$\liminf_{n\to\infty} \frac{1}{n}\ln \bar{\pi}_{n,k} \geq -\ln \alpha. \tag{8}$$

Finally, combining equations (6) and (8), we obtain that for each k,

$$\Lambda = \lim_{n\to\infty} \frac{1}{n}\ln \bar{\pi}_{n,k} = -\ln \alpha.$$

Together with the results on exact geometric decay in Li and Zhao (2005b) and Tai and Zhao (2010), we complete the proof. □

Remark 3.1. *It is noted that the case with $\alpha = \phi_{B_+}$ and $B_+^*(\alpha) < \infty$ is not covered by Theorem 3.1 since $B_+^*(z)$ is not meromorphic on $|z| \leq \alpha$. But, from Theorem 3.2 in Tai and Zhao (2010), if $\alpha = \phi_{B_+} < \phi_{A_+}$ and $B_+^*(\alpha) < \infty$, $\{\boldsymbol{\pi}_n, n = 0, 1, \ldots\}$ has exact geometric decay, and, consequently, Λ also exists.*

Next, we consider the case for which there exists no $|z| > 1$ such that $A^*(z)$ is finite and $\chi(z) = 1$. Before stating the results, we give a definition along the same line with Li and Zhao (2005a). For a sequence of matrices $\{M_n, n = 1, 2, \ldots\}$ of size $m\times m$, if there is a scalar sequence $\{\rho_n, n = 1, 2, \ldots\}$ and a finite matrix $W \gneq 0$ such that $\lim_{n\to\infty}\bar{M}_n/\bar{\rho}_n = W$, where $\bar{M}_n = \sum_{l=n}^{\infty} M_l$ and $\bar{\rho}_n = \sum_{l=n}^{\infty} \rho_l$, we call the sequence $\{\rho_n, n = 1, 2, \ldots\}$ and the matrix W a uniformly dominant sequence of the matrices $\{M_n, n = 1, 2, \ldots\}$ and the associated ratio matrix, respectively.

Theorem 3.2. *Assume that there exists no such $|z| > 1$ that $A^*(z)$ is finite and $\chi(z) = 1$. We also assume $\phi_{A_+} \leq \phi_{B_+}$. If $\{A_n, n = 1, 2, \ldots\}$ has a uniformly dominant sequence $\{\rho_n, n = 1, 2, \ldots\}$ with associated ratio matrix W satisfying $\liminf_{n\to\infty} \frac{1}{n}\ln \bar{\rho}_n \geq -\ln \phi_{A_+}$, then the decay rate in the logarithmic sense of $\{\boldsymbol{\pi}_n, n = 0, 1, \ldots\}$ along the level direction exists, and is given by*

$$\Lambda = -\ln \phi_{A_+}.$$

Proof: Without loss of generality, we assume that the (i_0, j_0)th entry of W is positive. Thus, the entry sequence $\{[A_n]_{i_0,j_0}, n = 1, 2, \ldots\}$ of $\{A_n, n = 1, 2, \ldots\}$ satisfies $\lim_{n\to\infty}[\bar{A}_n]{i_0,j_0}/\bar{\rho}_n = [W]_{i_0,j_0} > \frac{1}{2}[W]_{i_0,j_0}$, where $\bar{A}_n = \sum_{l=n}^{\infty} A_l$. Then, there exists sufficiently large N such that for all $n > N$,

$$[\bar{A}_n]_{i_0,j_0} \geq \frac{1}{2}[W]_{i_0,j_0}\bar{\rho}_n,$$

which leads to

$$\frac{1}{n}\ln[\bar{A}_n]_{i_0,j_0} \geq \frac{1}{n}\ln\left(\frac{1}{2}[W]_{i_0,j_0}\right) + \frac{1}{n}\ln \bar{\rho}_n.$$

Therefore, $\liminf_{n\to\infty} \frac{1}{n}\ln [\bar{A}_n]{i_0,j_0} \geq -\ln \phi_{A_+}$. Next, from equation (4), we know

$$\sum_{l=n}^{\infty} \boldsymbol{\pi}_l \geq \boldsymbol{\pi}_1 A_{n-1} + \boldsymbol{\pi}_1 \bar{A}_n \geq \boldsymbol{\pi}_1 \bar{A}_n,$$

which leads to

$$\frac{1}{n}\ln \bar{\pi}_{n,j_0} \geq \frac{1}{n}\ln \pi_{1,i_0} + \frac{1}{n}\ln[\bar{A}_n]_{i_0,j_0}.$$

Hence,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n}\ln \bar{\pi}_{n,j_0} &\geq \liminf_{n\to\infty} \frac{1}{n}\ln \pi_{1,i_0} \\ &+\liminf_{n\to\infty} \frac{1}{n}\ln[\bar{A}_n]_{i_0,j_0} \\ &= \liminf_{n\to\infty} \frac{1}{n}\ln[\bar{A}_n]_{i_0,j_0} \\ &\geq -\ln \phi_{A_+}.\end{aligned}$$

By Lemma 2.1, we obtain $\phi_{A_+} = \phi_R$. Since there exists no such $|z| > 1$ that $A^*(z)$ is finite and $\chi(z) = 1$, we have that $(I-R^*(z))^{-1}$ always exists when $1 < |z| < \phi_{A_+}$. Hence, by equation (5), we have that, for any $|z| < \phi_{A_+}$, $\pi^*(z)$ is finite since $\phi_{A_+} \leq \phi_{B_+}$, and, for any $|z| > \phi_{A_+}$, $\pi^*(z)$ is infinite since $R^*(z)$ is infinite. So ϕ_{A_+} is the radius of convergence of $\pi^*(z)$. Then we have by Lemma 2.4 (Cauchy-Hadamard theorem) that

$$\limsup_{n\to\infty} \frac{1}{n}\ln \bar{\pi}_{n,j_0} = -\ln \phi_{A_+}.$$

Consequently, we have

$$\lim_{n\to\infty} \frac{1}{n}\ln \bar{\pi}_{n,j_0} = -\ln \phi_{A_+}.$$

From Lemma 2.3, for every background state j,

$$\lim_{n\to\infty} \frac{1}{n}\ln \bar{\pi}_{n,j} = -\ln \phi_{A_+}.$$ □

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We wish to provide a short note on conditions of Theorem 3.2. In fact, that $\{A_n, n = 1, 2, \dots\}$ has a uniformly dominant sequence $\{\rho_n, n = 1, 2, \dots\}$ with associated ratio matrix W satisfying $\liminf_{n\to\infty} \frac{1}{n} \ln \bar{\rho}_n \geq -\ln \phi_{A_+}$ is equivalent to the condition that there exists an entry sequence, say $\{[A_n]_{i_0,j_0}, n = 1, 2, \dots\}$, of $\{A_n, n = 1, 2, \dots\}$ satisfying $\liminf_{n\to\infty} \frac{1}{n} \ln [\bar{A}_n]_{i_0,j_0} \geq -\ln \phi_{A_+}$.

In Theorems 3.1 and 3.2, we condition the logarithmic decay on the meromorphic function and the size of jumps of the Markov chain. In the following theorem, we provide a condition on the fundamental matrix. In the proof of the next theorem, we do not distinguish whether or not there exists $|z| = \alpha > 1$ such that $A^*(z)$ and $\chi(z) = 1$, since we can prove the theorem in a unified way.

Theorem 3.3. *If $\alpha \leq \phi_{B_+}$ (or $\phi_{A_+} \leq \phi_{B_+}$, instead, if α does not exist) where α is given in Theorem 3.1 if it exists, and Q_1 is irreducible, then the decay rate in the logarithmic sense of $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ along the level direction exists, and is given by*

$$\Lambda = -\ln \alpha \ (or {-}\ln \phi_{A_+}, instead, if\ \alpha\ does\ not\ exist).$$

Proof: Let $\hat{R}_{j,j}(n) = [\sum_{l=n}^{\infty} \sum_{u=1}^{\infty} R_l^{\circledast u}]_{j,j}$, where $R_l^{\circledast u}$ denotes u-fold convolution of $\{R_n, n = 1, 2, \dots\}$ (see Li and Zhao (2005a)). Then $\hat{R}_{j,j}(l+n) \geq \hat{R}_{j,j}(l)\hat{R}_{j,j}(n)$. First, we show for some j, the sequence $\{\hat{R}_{j,j}(n), n = 1, 2, \dots\}$ satisfies Lemma 2.2 if Q_1 is irreducible. Then a lower bound for the lower decay rate in the logarithmic sense of $\{\pi_{n,j}, n = 0, 1, \dots\}$ can be obtained in terms of the convergent parameter of $\{\hat{R}_{j,j}(n), n = 1, 2, \dots\}$ in the next two parts.

Part 1: Let $\mathbf{e}$ be the column vector of ones. Firstly, if $\sum_{n=2}^{\infty} A_n \boldsymbol{e} \neq \mathbf{0}$, let $L_k(\bar{A}_2)$ be the set of states in L_k (i.e., level k) corresponding to positive elements of the vector $\sum_{n=2}^{\infty} A_n e$, and $\tilde{L}_k(\bar{A}_2)$ be the set of states in L_k corresponding to zero elements of the vector $\sum_{n=2}^{\infty} A_n \boldsymbol{e}$. Let $\bar{R}(2) = \sum_{n=2}^{\infty} R_n$, by Lemma 4 in Zhao (2000),

$$\bar{R}(2) = \left(\sum_{n=2}^{\infty} A_n, \sum_{n=2}^{\infty} A_{n+1}, \sum_{n=2}^{\infty} A_{n+2}, \cdots \right) \hat{Q}_1(\cdot, 1),$$

where $\hat{Q}_1(\cdot, 1)$ is the first column of the matrix $\hat{Q}_1$. Therefore, the jth row of $\bar{R}(2)$ is a zero vector if $(k, j) \notin L_k(\bar{A}_2)$. Let $R_{\bar{A}_2}$ be the sub-matrix of $\bar{R}(2)$ corresponding to the index set $L_k(\bar{A}_2) \times L_k(\bar{A}_2)$. If Q_1 is irreducible, $R_{\bar{A}_2}$ is a positive matrix and, consequently, irreducible since $\hat{Q}_1(\cdot, 1)$ is positive. So, we can find an irreducible subclass of $\bar{R}(2)$. By suitably changing the order of rows and columns, $\bar{R}(2)$ can be rewritten as

$$\bar{R}(2) = (R_{\bar{A}_2} \tilde{R}_{\bar{A}_2} 00),$$

where $\tilde{R}_{\bar{A}_2}$ is the submatrix of $\bar{R}(2)$ corresponding to the index set $L_k(\bar{A}_2) \times \tilde{L}_k(\bar{A}_2)$. Therefore, we can find a positive element on the main diagonal of $\bar{R}(2)$, that is, for some j, the sequence $\{\hat{R}_{j,j}(n), n = 1, 2, \dots\}$ satisfies Lemma 2.2. In fact, the sequence $\{\hat{R}_{j,j}(n), n = 1, 2, \dots\}$ is decreasing. If for some n_0, $\hat{R}_{j,j}(n_0)$ is positive, then for all $n \leq n_0$, $\hat{R}_{j,j}(n)$ is positive. Hence, the set of those integers $n \geq 2$ for which $\hat{R}_{j,j}(n) < 0$ has $g.c.d.$ of 1. The set is non-empty since $\hat{R}_{j,j}(2) > 0$.

Secondly, if $\sum_{n=1}^{\infty} A_n \boldsymbol{e} \neq \mathbf{0}$ and $\sum_{n=2}^{\infty} A_n \boldsymbol{e} = \sum_{n=3}^{\infty} A_n \boldsymbol{e} = \cdots = \mathbf{0}$, then we have $R_1 \neq 0$, $R_2 = R_3 = \cdots = 0$. Hence, we have the following representations:

$$\boldsymbol{\pi}^*(z) = \boldsymbol{\pi}_0 R_0^*(z)(I + zR_1 + z^2 R_1^2 + \cdots + z^n R_1^n + \cdots).$$

We can find a positive element on the main diagonal of R_1. Therefore, we can similarly apply Lemma 2.2 to R_1^n.

Thirdly, if $\sum_{n=1}^{\infty} A_n \boldsymbol{e} = \sum_{n=2}^{\infty} A_n \boldsymbol{e} = \sum_{n=3}^{\infty} A_n \boldsymbol{e} = \cdots = \mathbf{0}$, then we have $R_1 = R_2 = R_3 = \cdots = 0$ and $\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 R_{0,n}$. It is impossible when non-boundary transition probabilities play a dominant role in tail behavior.

Part 2: We construct a lower bound for the tail of the stationary distribution in terms of $\{\hat{R}_{j,j}(n), n = 1, 2, \dots\}$. By the discussion in the first part, we know that there is at least one positive entry on the main diagonal of $\bar{R}(2)$ (or R_1 in the second case, instead). Without loss of generality, we assume that the (j, j)th entry on the main diagonal of $\bar{R}(2)$ (or R_1 in the second case, instead) is positive. From equation (3.8) in Li and Zhao (2005a), we have that

$$\bar{\boldsymbol{\pi}}_n = \sum_{l=n}^{\infty} \boldsymbol{\pi}_0 R_{0,l} \circledast \sum_{u=1}^{\infty} R_l^{\circledast u}. \tag{9}$$

Since there is at least one positive entry in each column of $R_0 \triangleq R_0^*(1)$, we assume without loss of generality that the positive entry of the jth column of R_0 is $[R_0^*(1)]_{i_0,j} = [R_{0,1}]_{i_0,j} + [R_{0,2}]_{i_0,j} + \dots$, where $[R_{0,n}]_{i_0,j}$ is the (i_0, j)th entry of $R_{0,n}$. We can find a sufficiently large N such that for some $n_0 < N$, $[R_{0,n_0}]_{i_0,j} > 0$, since the tail of $[R_0^*(1)]_{i_0,j}$ tends to 0. From equation (9), we have that for sufficiently large n,

$$\bar{\boldsymbol{\pi}}_n \geq \boldsymbol{\pi}_0 R_{0,n_0} \left[\sum_{l=n}^{\infty} \sum_{u=1}^{\infty} R_{l-n_0}^{\circledast u} \right] \qquad \left(\text{or} \quad \boldsymbol{\pi}_0 R_{0,n_0} \sum_{l=n}^{\infty} R_1^{l-n_0} \right).$$

Then,

$$\bar{\pi}_{n,j} \geq \pi_{0,i_0} [R_{0,n_0}]_{i_0,j} \left[\sum_{l=n}^{\infty} \sum_{u=1}^{\infty} R_{l-n_0}^{\circledast u} \right]_{j,j}. \tag{10}$$

Part 3: We analyze the convergent parameter of $\{\hat{R}_{j,j}(n), n = 1, 2, \dots\}$. If there exists an $|z| = \alpha > 1$ such that $\chi(z) = 1$ and $A^*(z)$ is finite, we have that α is the radius of convergence of $\sum_{n=0}^{\infty} z^n \sum_{u=1}^{\infty} R_n^{\circledast u}$. In fact, for $1 < |z| < \alpha$, $\sum_{n=0}^{\infty} z^n \sum_{u=1}^{\infty} R_n^{\circledast u} = (I - R^*(z))^{-1}$ is finite, but $\sum_{n=0}^{\infty} \alpha^n \sum_{u=1}^{\infty} R_n^{\circledast u}$ is infinite since

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the inverse of $I–R^*(\alpha)$ does not exist. If there exists no $|z| > 1$ such that $A^*(z)$ is finite and $\chi(z) = 1$, we have that ϕ_{A_+} is the radius of convergence of $\sum_{n=0}^{\infty} z^n \sum_{u=1}^{\infty} R_n^{\circledast u}$, instead. By Remark 2.1, α (or ϕ_{A_+}, instead, if α does not exist) is the convergent parameter of $\{\hat{R}_{j,j}(n), n = 1, 2, \dots\}$. Therefore, from Lemma 2.2 and equation (10), we have

$$\liminf_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,j} \geq -\ln \alpha (\text{or} -\ln \phi_{A_+}, \text{instead, if } \alpha \text{ does not exist}).$$

Since α (or ϕ_{A_+}, instead, if α does not exist) is the radius of convergence of $\boldsymbol{\pi}^*(z)$ (see corresponding discussion in Theorem 3.1 and 3.2 for details), by Cauchy-Hadamard theorem, we have

$$\limsup_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,j} = -\ln \alpha (\text{or} -\ln \phi_{A_+}, \text{instead, if } \alpha \text{ does not exist}).$$

So,

$$\lim_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,j} = -\ln \alpha (\text{or} -\ln \phi_{A_+}, \text{instead, if } \alpha \text{ does not exist}).$$

Then, from Lemma 2.3, we obtain that

$$\Lambda = -\ln \alpha \ (\text{or} -\ln \phi_{A_+}, \text{instead, if } \alpha \text{ does not exist}). \quad \square$$

If $\phi_{B_+} < \alpha$ (or $\phi_{B_+} < \phi_{A_+}$ if α does not exist), boundary transition probabilities have a dominant impact on tail behavior. In the following, we provide conditions on the logarithmic decay in this situation.

Theorem 3.4. *If $\phi_{B_+} \leq \alpha$ (or $\phi_{B_+} \leq \phi_{A_+}$, instead, if α does not exist), where α is given in Theorem 3.1 if it exists, and $\{B_n, n = 1, 2, \dots\}$ has a uniformly dominant sequence $\{\rho_n, n = 1, 2, \dots\}$ with associated ratio matrix W satisfying $\liminf_{n\to\infty} \frac{1}{n} \ln \bar{\rho}_n \geq -\ln \phi_{B_+}$, then the decay rate in the logarithmic sense of $\{\boldsymbol{\pi}_n, n = 0, 1, \dots\}$ along the level direction exists, and is given by*

$$\Lambda = -\ln \phi_{B_+}.$$

Proof: Without loss of generality, we assume that the (i_0, j_0)th entry of W is positive. Therefore, the entry sequence $\{[B_n]_{i_0,j_0}, n = 1, 2, \dots\}$ of $\{B_n, n = 1, 2, \dots\}$ satisfies $\lim_{n\to\infty} [\bar{B}_n]_{i_0,j_0} / \bar{\rho}_n = [W]_{i_0,j_0} > \frac{1}{2}[W]_{i_0,j_0}$, where $\bar{B}_n = \sum_{l=n}^{\infty} B_l$. Hence, $\liminf_{n\to\infty} \frac{1}{n} \ln[\bar{B}_n]i_0, j_0 \geq -\ln \phi_{B_+}$. From equation (4) and Theorem 12 in Zhao (2000), we have that

$$\boldsymbol{\pi}_n \geq \boldsymbol{\pi}_0 R_{0,n} \geq \boldsymbol{\pi}_0 B_n.$$

Then,

$$\bar{\pi}_{n,j_0} \geq \pi_{0,i_0} [\bar{B}_n] i_0, j_0,$$

which leads to

$$\frac{1}{n} \ln \bar{\pi}_{n,j_0} \geq \frac{1}{n} \ln \pi_{0,i_0} + \frac{1}{n} \ln [\bar{B}_n]_{i_0,j_0}.$$

Hence,

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,j_0} &\geq \liminf_{n\to\infty} \frac{1}{n} \ln \pi_{0,i_0} \\ &\quad + \liminf_{n\to\infty} \frac{1}{n} \ln [\bar{B}_n]_{i_0,j_0} \\ &= \liminf_{n\to\infty} \frac{1}{n} \ln [\bar{B}_n]_{i_0,j_0} \\ &\geq -\ln \phi_{B_+}. \end{aligned}$$

On the other hand, we assume that $\phi_{B_+} \leq \alpha$ if α exists. In this case, ϕ_{B_+} is the radius of convergence of $\pi^*(z)$. In fact, we have that, for any $1 < |z| < \phi_{B_+}$, $R_0^*(z)$ is finite and the inverse of $I–R^*(z)$ exists, therefore, $\pi^*(z)$ is finite. But, for $|z| > \phi_{B_+}$, $\pi^*(z)$ is infinite since $R_0^*(z)$ is infinite. If α does not exist, we assume that $\phi_{B_+} \leq \phi_{A_+}$, instead. Similarly, for any $1 < |z| < \phi_{B_+}$, $R_0^*(z)$ is finite and the inverse of $I–R^*(z)$ exists, therefore, $\pi^*(z)$ is finite; for $|z| > \phi_{B_+}$, $\pi^*(z)$ is infinite, since $R_0^*(z)$ is infinite. Thus, ϕ_{B_+} is the radius of convergence of $\pi^*(z)$. By Lemma 2.4 (Cauchy-Hadamard theorem), we have

$$\limsup_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,j_0} = -\ln \phi_{B_+}.$$

Then,

$$\lim_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,j_0} = -\ln \phi_{B_+}.$$

From Lemma 2.3, for each background state j, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,j} = -\ln \phi_{B_+}. \quad \square$$

Theorem 3.5. *If $\phi_{B_+} < \alpha (\text{or} \phi_{B_+} < \phi_{A_+} \text{if} \alpha \text{ does not exist})$, where α is given in Theorem 3.1 if it exists, and $B_+^*(z)$ is meromorphic on $|z| \leq \phi_{B_+}$, then the decay rate in the logarithmic sense of $\{\pi_n, n = 0, 1, \dots\}$ along the level direction exists, and is given by*

$$\Lambda = -\ln \phi_{B_+}.$$

Proof: We show that ϕ_{B_+} is the radius of convergence of $\pi^*(z)$. Since $R_0^*(z)$ is finite on $|z| < \phi_{B_+}$ and $(I–R^*(z))^{-1}$ exists on $1 < |z| \leq \phi_{B_+}$, we have that $\pi^*(z)$ is finite on $|z| < \phi_{B_+}$. Due to $\pi^*(\phi_{B_+}) \geq \pi_0 R_0^*(\phi_{B_+}) \geq \pi_0 B_+^*(\phi_{B_+}) = \infty$, we obtain that ϕ_{B_+} must be the radius of convergence of $\pi^*(z)$. Then, from Lemma 2.4 (Cauchy-Hadamard theorem), we have that for some k_0,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \bar{\pi}_{n,k_0} = -\ln \phi_{B_+}.$$

Lemma 2.3 implies the above inequality holds when k_0 is any background state. Similar to the proof of Theorem 3.1, we have that $-\ln\phi_{B_+}$ is also a lower bound on $\liminf_{n\to\infty}\frac{1}{n}\ln\bar{\pi}_{n,k}$, for each k. Hence, the decay rate in the logarithmic sense of $\{\pi_n, n=0,1,\ldots\}$ along the level direction exists, and is given by

$$\Lambda = -\ln\phi_{B_+}. \qquad \square$$

Remark 3.2. *The decay rate in the logarithmic sense may not exist. A counter example was given in Nakagawa (2004) showing that the stationary distribution does not decay exponentially.*

4. APPLICATIONS TO THE $BMAP/G/1$ QUEUE WITH MULTIPLE VACATIONS

In this section, we apply the theoretical results to a single server exhaustive multiple vacation queue with a batch Markovian arrival process (*BMAP*). The $BMAP/G/1$ queue with multiple vacations is defined as follows.

- Customers arrive according to a *BMAP* with matrix representation $(D_0, D_1, \ldots)$, where $\{D_k, k=0,1,\ldots\}$ are $m\times m$ matrices, D_0 has negative diagonal elements and non-negative off-diagonal elements, and $\{D_k, k=1,2,\ldots\}$ are non-negative. Assume that $D \triangleq \sum_{k=0}^{\infty} D_k$ is an irreducible infinitesimal generator and $D \neq D_0$. The fundamental arrival rate is defined as $\lambda = \boldsymbol{\theta}\sum_{n=1}^{\infty} nD_n e$, where $\boldsymbol{\theta}$ satisfies $\boldsymbol{\theta}D=0$ and $\boldsymbol{\theta e}=\mathbf{1}$.
- There is a single server in the system. Service times are i.i.d. random variables with distribution function $W(t)$ and finite mean $1/\mu$;
- We assume that a vacation of the sever begins when the system becomes empty. At a vacation completion instant, if there is no customer in the system, then the server takes another vacation and repeats it until there is at least one customer in the system at the end of a vacation; For the latter case, the server begins to serve customers.
- Vacation times are i.i.d. random variables with distribution function $V(t)$ and finite mean $1/\nu$.
- The arrival process, the service times and the vacation times are all assumed to be mutually independent.

We now turn to the study of the Markov chain associated with the queue length process embedded at departure epoches for the $BMAP/G/1$ queue with multiple vacations, whose transition matrix is of the $M/G/1$-type and is given by

$$P = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ A_{-1} & A_0 & A_1 & A_2 & \cdots \\ & A_{-1} & A_0 & A_1 & \cdots \\ & & A_{-1} & A_0 & \cdots \\ & & & \ddots & \ddots \end{pmatrix},$$

where $\{B_n, n=0,1,\ldots\}$ and $\{A_n, n=-1,0,1,\ldots\}$ are $m\times m$ matrices, which are defined as

$[A_n]_{i,j} = \mathbb{P}\{$Given a departure at time 0, which left at least one customer in the system and the arrival process in phase i, the next departure occurs with the arrival process in phase j, and during that service there were $n+1$ arrivals$\}$.

$[B_n]_{i,j} = \mathbb{P}\{$Given a departure at time 0, which left the system empty and the arrival process in phase i, the next departure occurs with the arrival process in phase j, leaving n customers in the system$\}$.

We similarly define $\{V_n, n=0,1,\ldots\}$ as follows

$[V_n]_{i,j} = \mathbb{P}\{$Given that a vacation begins at time 0, with the arrival process in phase i, the end of vacation occurs with the arrival process in phase j, and during the vacation there were n arrivals$\}$.

We assume that $\rho = \frac{\lambda}{\mu} < 1$ so that the queueing system is stable and the Markov chain is ergodic. Define generating functions

$$A^*(z) = \sum_{n=-1}^{\infty} z^n A_n, \qquad B^*(z) = \sum_{n=0}^{\infty} z^n B_n,$$
$$V^*(z) = \sum_{n=0}^{\infty} z^n V_n, \qquad D^*(z) = \sum_{n=0}^{\infty} z^n D_n.$$

Without loss of generality, we only consider z in real number in this section. By analogy to Lucantoni et al. (1990), we have the following properties

$$zA^*(z) = \int_0^{\infty} e^{D^*(z)t}dW(t), \quad V^*(z) = \int_0^{\infty} e^{D^*(z)t}dV(t). \quad (11)$$

From Theorem 1 in Lucantoni et al. (1990),

$$B^*(z) = [I - V^*(0)]^{-1}[V^*(z) - V^*(0)]A^*(z). \quad (12)$$

Define $\phi_V = \min_{1\le i,j\le m}\sup\{z\ge 1 : [V^*(z)]_{i,j} < \infty\}$. Recall that ϕ_{A_+} and ϕ_{B_+} are radii of convergence of $A^*(z)$ and $B^*(z)$, respectively. In our study, we assume that $W(t)$ and $V(t)$ are light-tailed. From equation (11), $\{A_n, n=-1,0,1,\ldots\}$ being light-tailed is equivalent to $W(t)$ being light-tailed; and $\{V_n, n=0,1,\ldots\}$ being light-tailed is equivalent to $V(t)$ being light-tailed. For $z>0$, let $\rho(z)$ the eigenvalue with the greatest real part of $D^*(z)$ and $\tilde{\gamma}(z)$ be the Perron-Frobenius eigenvalue of $zA^*(z)$. Set $\boldsymbol{\mu}(z)$ and $\boldsymbol{\nu}(z)$ to be the left and right eigenvectors corresponding to $\rho(z)$, respectively, satisfying $\boldsymbol{\mu}(z)\boldsymbol{e}=1$ and $\boldsymbol{\mu}(z)\boldsymbol{\nu}(z)=1$. Then, the following properties hold:

(a) For $z>0$, we have that $\rho(z)$ is strictly increasing in z (see M-matrix in Seneta (1981)).

(b) $\boldsymbol{\mu}(z)$ and $\boldsymbol{\nu}(z)$ are also the left and right eigenvectors of $zA^*(z)$ corresponding to $\tilde{\gamma}(z)$, respectively.

(c) $\tilde{\gamma}(z) = \int_0^{\infty} e^{\rho(z)t}dW(t)$.

Lemma 4.1. *There exists sufficiently large z_0 such that $\rho(z) > z^{\frac{m_0}{m}}$ and $\tilde{\gamma}(z) \ge z$, where $1\le m_0\le m$, for $z>z_0$.*

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Proof: Choose $\delta \geq \max\{-d_{i,i}, i = 1, 2, \ldots, m\}$, where $\{d_{i,i}, i = 1, 2, \ldots, m\}$, are the diagonals of D_0. Then, $D^*(z) + \delta I \geq 0$ and $\rho(z) + \delta$ is the Perron-Frobonius eigenvalue of $D^*(z) + \delta I$. Since D is irreducible and $D_0 \neq D$, $D^*(z)$ is irreducible and we assume without loss of generality that $D_1 \neq 0$. Therefore, $\rho(z) + \delta$ is strictly increasing in z. then

$$\begin{aligned}(\rho(z)+\delta)^m &\geq |det(D^*(z)+\delta I)| \\ &= |c_{m_0} z^{m_0} + c_{m_0-1} z^{m_0-1} + \ldots + c_0|, \\ &\quad (\text{w.l.o.g.}, c_m = \ldots = c_{m_0+1} = 0, c_{m_0} \neq 0) \\ &= |c_{m_0}||z^{m_0}| \left| 1 + \frac{c_{m_0-1}}{c_{m_0}} \frac{1}{z} + \ldots + \frac{c_0}{c_{m_0}} \frac{1}{z^{m_0}} \right|.\end{aligned}$$

If $z \to \infty$, then $(\rho(z) + \delta)^m \geq (|c_{m_0}| + 1)|z^{m_0}|$. Therefore, $\rho(z) \geq \sqrt[m]{|c_{m_0}| + 1} z^{\frac{m_0}{m}} - \delta$ if z is sufficiently large. Since the service time is not zero with probability one, there exists $t_0 > 0$ such that $W(t_0) < 1$, or equivalently, $1 - W(t_0) > 0$. Hence, we have, for sufficiently large z,

$$\begin{aligned}\tilde{\gamma}(z) &\geq \int_{t_0}^{\infty} e^{\rho(z)t} dW(t) \\ &\geq (1 - W(t_0)) e^{\rho(z)t_0} \\ &\geq (1 - W(t_0)) e^{(\sqrt[m]{|c_{m_0}|+1} z^{\frac{m_0}{m}} - \delta)t_0} \\ &\geq z.\end{aligned}$$

□

Lemma 4.1 provides an approach to find $1 < \alpha < \phi_{A_+}$ such that $\tilde{\gamma}(\alpha) = \alpha$. Let $\tilde{W}(s)$ be the Laplace-Stieltjes transform of $W(t)$ and σ_W the abscissa of convergence of $\tilde{W}$, that is, if $Re(s) < \sigma_W$, $\tilde{W}(s)$ diverges; $Re(s) > \sigma_W$, $\tilde{W}(s)$ converges. Then, $\phi_{A_+} = \rho^{-1}(-\sigma_W)$ and $\phi_{A_+} > 1 \Leftrightarrow \sigma_W < 0$ (Falkenberg 1994). We also have that

$$\tilde{\gamma}(z) = z, z \in (1, \phi_{A_+}] \Leftrightarrow \tilde{W}(x) = \rho^{-1}(-x), x \in [\sigma_W, 0).$$

Then, we have the following lemma.

Lemma 4.2. *A solution to equation* $\tilde{W}(x) = \rho^{-1}(-x)$, $x \in [\sigma_W, 0)$, *exists if and only if* $\lim_{x \to \sigma_W} \tilde{W}(x) \geq \rho^{-1}(-\sigma_W)$.

Proof: Sufficiency: If $\tilde{W}$ is divergent at $\sigma_W < 0$ or $\tilde{W}$ is convergent at $\sigma_W < 0$ and $\lim_{x \to \sigma_W} \tilde{W}(x) > \rho^{-1}(-\sigma_W)$, it is confirmed in Falkenberg (1994). If $\tilde{W}$ is convergent at $\sigma_W < 0$ and $\lim_{x \to \sigma_W} \tilde{W}(x) = \rho^{-1}(-\sigma_W)$, we have that the solution is $x_0 = \sigma_W = -\rho(\phi_{A_+})$.

Necessity: It is obvious since there is at most a solution to equation: $\tilde{\gamma}(z) = z, z \in (1, \phi_{A_+}]$. □

If α exists, it is clear that $1 < \alpha \leq \phi_{A_+}$. It is possible that α does not exists (See Example 4.2). For this case, we assume $\alpha = \infty$ for convenience. By equation (12), we have that $\phi_{B_+} = \min\{\phi_{A_+}, \phi_V\}$. Define $\hat{\rho} = (\min\{\alpha, \phi_{A_+}, \phi_V\})^{-1}$. Now, we are ready to state and prove the main result of this section.

Theorem 4.1. *Assume that $W(t)$ and $V(t)$ are light-tailed. If the logarithmic asymptotic of the stationary distribution for the $BMAP/G/1$ vacation queue exists, then the decay rate in the logarithmic sense is given by*

$$\Lambda = \ln\hat{\rho}.$$

Furthermore, if exact geometric decay exists, then the decay rate is given by

$$\hat{\rho} = \frac{1}{\min\{\alpha, \phi_V\}}.$$

Proof: By an analogous argument to Section 3, we have that $1/\hat{\rho}$ is the radius of convergence of $\pi^*(z)$. By Cauchy-Hadamard theorem, $\bar{\Lambda} = \ln\hat{\rho}$. Therefore, if the logarithmic asymptotic of the stationary distribution for the $BMAP/G/1$ vacation queue exists, the decay rate in the logarithmic sense is given by $\Lambda = \ln\hat{\rho}$. □

In the following, we present some interesting observations on the relationship between the vacation time and the tail of the stationary distribution of the queue length. Methods for computing ϕ_{A_+}, ϕ_V and α are provided as well.

Example 4.1. The $M/M/1$ queue with multiple vacations In this queueing model, customers arrive according to a Poisson process with parameter λ, the service time has an exponential distribution with parameter μ, and the vacation time has a distribution $V(t)$. For this example, we have

$$zA^*(z) = \frac{\mu}{\lambda + \mu - \lambda z}, \qquad V^*(z) = \int_0^{\infty} e^{-\lambda(1-z)t} dV(t).$$

It is easy to find $\alpha = \frac{\mu}{\lambda}$ and $\phi_{A_+} = 1 + \frac{\mu}{\lambda}$, respectively. From Theorem 4.1, the decay rate $\hat{\rho} = (\min\{\alpha, \phi_V\})^{-1}$, in the logarithmic sense. Next, we find explicit solutions for a series of cases with specific distributions of the vacation time.

a) ***$V(t)$* has an exponential distribution with parameter *ν*** For this case, $V^*(z) = \int_0^{\infty} e^{-\lambda(1-z)t} \nu e^{-\nu t} dt = \frac{\nu}{\lambda + \nu - \lambda z}$. It is easy to find $\phi_V = 1 + \frac{\nu}{\lambda}$. Then we have that (1) if $\mu < \lambda + \nu$, the exact geometric decay rate is given by $\hat{\rho} = \frac{\lambda}{\mu}$; (2) if $\mu = \lambda + \nu$, the decay rate in the logarithmic sense is given by $\ln\hat{\rho}$ with $\hat{\rho} = \frac{\lambda}{\mu}$; and (3) if $\mu > \lambda + \nu$, the exact geometric decay rate is given by $\hat{\rho} = \frac{\lambda}{\lambda + \nu}$.

This example indicates that, if the vacation time is short, the decay rate of the distribution of the queue length remains the same, while the distribution of the queue length is different for different (small) $1/\nu$. So, for cases with a short vacation time (i.e., small $1/\nu$), the tail asymptotics of the distribution of queue length is not changed. On the other hand, if the vacation time is long (i.e., large $1/\nu$), the decay rate of the distribution of the queue length changes as the distribuiton of the vacation time changes. This observation holds, in the logarithmic

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sense, for the general case as indicated by Theorem 4.1 and demonstrated by Figure 4.1.

b) ***$V(t)$ has an Erlang distribution (k,θ)*** *Consider* $V(t)$ with the probability density function $f(t,k,\theta)=t^{k-1}\frac{e^{-\frac{t}{\theta}}}{\theta^k\Gamma(k)}$, where $t\geq 0, k,\theta>0$ and k is an integer. In this case, we have

$$V^*(z)=\int_0^\infty e^{-\lambda(1-z)t}t^{k-1}\frac{e^{-\frac{t}{\theta}}}{\theta^k\Gamma(k)}dt=\frac{(\frac{1}{\theta})^k}{(\lambda(1-z)+\frac{1}{\theta})^k}.$$

It is easy to find $\phi_V=1+\frac{1}{\theta\lambda}$. Then we have that (1) if $\mu<\lambda+\frac{1}{\theta}$, the exact geometric decay rate is given by $\hat{\rho}=\frac{\lambda}{\mu}$; (2) if $\mu=\lambda+\frac{1}{\theta}$, the decay rate in the logarithmic sense is given by $\ln\hat{\rho}$ with $\hat{\rho}=\frac{\lambda}{\mu}$; and (3) if $\mu>\lambda+\frac{1}{\theta}$, the decay rate in the logarithmic sense is given by $\hat{\rho}=\lambda(\lambda+\frac{1}{\theta})^{-1}$. That is, $\hat{\rho}=\lambda(\min\{\mu,\lambda+\frac{1}{\theta}\})^{-1}$, for all cases.

We would like to remark that, for the $M/M/1$ queue with Erlang vacation times, the asymptotic results can be obtained by using the matrix-geometric solution $\boldsymbol{\pi}_n=\boldsymbol{\pi}_1R^{n-1}$, $n\geq 1$, for the queue length at an arbitrary time. For this case, the Jordan canonical form of the matrix R can be found explicitly. Consequently, the tail asymptotics can be identified. If $\mu<\lambda+\frac{1}{\theta}$, the Perron-Frobenius eigenvalue of R is simple and $\boldsymbol{\pi}_n\sim\hat{\rho}^{-n}$. If $\mu=\lambda+\frac{1}{\theta}$, the Perron-Frobenius eigenvalue of R is not simple and $\boldsymbol{\pi}_n\sim n^{k+1}\hat{\rho}^{-n}$. If $\mu>\lambda+\frac{1}{\theta}$, the Perron-Frobenius eigenvalue of R is not simple and $\boldsymbol{\pi}_n\sim n^k\hat{\rho}^{-n}$. However, the approach does not work for more general cases (See Example 4.3).

c) ***$V(t)$ has an Weibull distribution (k,θ)*** Consider $V(t)$ with the probability density function $f(t,k,\theta)=\frac{k}{\theta}(\frac{t}{\theta})^{k-1}e^{-(\frac{t}{\theta})^k}$, where $t\geq 0, k,\theta>0$. The mean of $V(t)$ is $\theta\Gamma(1+\frac{1}{k})$. We have that $V_n=\int_0^\infty\frac{(\lambda t)^n}{n!}e^{-\lambda t}\frac{k}{\theta}(\frac{t}{\theta})^{k-1}e^{-(\frac{t}{\theta})^k}dt$, therefore, $V^*(z)=\int_0^\infty e^{-\lambda(1-z)t}\frac{k}{\theta}(\frac{t}{\theta})^{k-1}e^{-(\frac{t}{\theta})^k}dt$ For $k<1$, Weibull distribution is heavy-tailed. Therefore, we have $\phi_V=1$. We see from equation (12) that when the vacation time has a heavy-tailed distribution, the stationary distribution is also heavy-tailed.

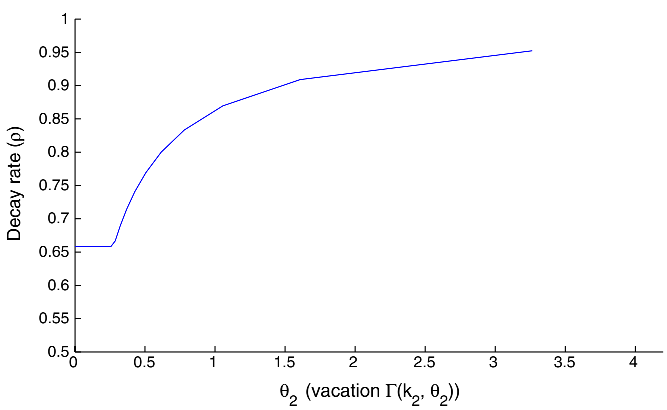

Figure 1. The decay rate in logarithmic sense

This example shows that the tail of the distributin of the queue length can be affected greatly by the vacation time distribution, not just by the mean and the variance of the vacation time. In fact, for this case, the queue length distribution is heavy-tailed no matter what is the mean and variance of the vacation time (as long as system stabiltiy is guaranteed.)

Example 4.2. The $M/G/1$ queue with multiple vacations Assume that the arrival process is Poisson with parameter λ, the service time has distribution $W(t)$, and the vacation time has distribution $V(t)$. We also assume that there exists $|z_0|>1$ such that $V^*(z_0)=\int_0^\infty e^{-\lambda(1-z_0)t}dV(t)<\infty$, that is, $V(t)$ is light-tailed. Now, we consider $W(t)$ as $W(t)=0$ for $t<1$; $W(t)=f_0+\int_1^t\frac{1}{x^2}e^{-\mu x}dx$ for $t\geq 1$, where $\mu>0$, for which Falkenberg (1994) showed that α does not exist if $0<\lambda<\frac{\mu}{f_0e^\mu}$. Then, if the logarithmic asymptotic exists, the decay rate in the logarithmic sense is given by $\ln\hat{\rho}$ with $\hat{\rho}=(\min\{\phi_V,\phi_{A_+}\})^{-1}$.

Example 4.3. The *BMAP/Gamma/1* **queue with multiple vacations** Consider $W(t)$ with the probability density function $f(t,k_1,\theta)=t^{k_1-1}\frac{e^{-\frac{t}{\theta_1}}}{\theta_1^{k_1}\Gamma(k_1)}$, where $t\geq 0, k_1,\theta_1>0$, and $V(t)$ with the probability density function $f(t,k_2,\theta_2)=t^{k_2-1}\frac{e^{-\frac{t}{\theta_2}}}{\theta^{k_2}\Gamma(k_2)}$, where $t\geq 0, k_2,\theta_2>0$. We have

$$zA^*(z)=\int_0^\infty e^{D^*(z)t}t^{k_1-1}\frac{e^{-\frac{t}{\theta_1}}}{\theta^{k_1}\Gamma(k_1)}dt,\quad\text{and}$$

$$V^*(z)=\int_0^\infty e^{D^*(z)t}t^{k_2-1}\frac{e^{-\frac{t}{\theta_2}}}{\theta^{k_2}\Gamma(k_2)}dt,$$

respectively. Therefore, we get

$$\boldsymbol{\mu}(z)zA^*(z)\nu(z)=\int_0^\infty e^{\rho(z)t}t^{k_1-1}\frac{e^{-\frac{t}{\theta_1}}}{\theta^{k_1}\Gamma(k_1)}dt=\frac{1}{(1-\theta_1\rho(z))^{k_1}},$$

$$\boldsymbol{\mu}(z)V^*(z)\nu(z)=\int_0^\infty e^{\rho(z)t}t^{k_2-1}\frac{e^{-\frac{t}{\theta_2}}}{\theta^{k_2}\Gamma(k_2)}dt=\frac{1}{(1-\theta_2\rho(z))^{k_2}},$$

respectively. It is easy to know that ϕ_{A_+} can be solved from $1-\theta_1\rho(z)=0$ and ϕ_V from $1-\theta_2\rho(z)=0$. That α can be found by using the following computational method:

1) Take $\alpha_0=1$ and $\alpha_1=\phi_{A_+}$;
2) Let $\alpha_2=\frac{\alpha_0+\alpha_1}{2}$ and $\alpha_3=\frac{1}{(1-\theta_1\rho(\alpha_2))^{k_1}}$:
   If $\alpha_2>\alpha_3$, then $\alpha_0=\alpha_2$;
   If $\alpha_2<\alpha_3$, then $\alpha_1=\alpha_2$;
3) Repeat 2) until $|\alpha_1-\alpha_0|<\varepsilon$ (small constant). Then, $\alpha=\frac{\alpha_0+\alpha_1}{2}$.

From Theorem 4.1, if the logarithmic asymptotic exists, the decay rate is given by $\ln\hat{\rho}$ with $\hat{\rho}=(\min\{\alpha,1+\frac{1}{\lambda\theta_2}\})^{-1}$.

For the case with a *Gamma* vacation time, we have $E[V]=k_2\theta_2$ and $Var(V)=k_2\theta_2^2$. For the *BMAP/Gamma/1* queue with multiple vacations, the decay rate $\hat{\rho}$, as a function of θ_2, is plotted

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in Figure 1. As is shown in Figure 1, the decay rate is constant for small θ_2 (which implies a small mean vacation time). When θ_2 is large, the decay rate changes along with θ_2. It is interesting to note that, if k_2 changes, the decay rate (in logarithmic sense) $\hat{\rho}$ remains the same and yet the tail asymptotics changes as shown by part b) of Example 4.1. Note that for the Gamma distribution, k_2 can be any positive real number. Since the mean vacation time is changing with k_2, this example shows that the decay rate in logarithmic sense may or may not change at all when the mean vacation time changes. The same observation holds for the variance of the vacation time as well.

Finally, we like to point out that similar results can be obtained for $BMAP/G/1$ queues with a single vacation. For this case, $A^*(z)$ remains the same, but $B^*(z)$ is changed to

$$B^*(z) = [V^*(z) - V^*(0)]A^*(z).$$

It is easy to see that the convergence radius of $B^*(z)$ is again given by $\min\{\phi_{A_+}, \phi_V\}$. Consequently, Theorem 4.1 and the above discussion hold for this case.

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REFERENCES

Abate, J., Choudhury, G., and Whitt, W. (1994), “Asymptotics for steady-state tail probabilities in structured Markov queueing models”, *Stochastic Models,* 10: 99–143.

Abate, J., Choudhury, G., and Whitt, W. (1995), ““Exponential approximations for tail probabilities of queues”, I: waiting times”, *Operations Research,* 43(5): 885–901.

Alsmeyer, G. (1994), “On the Markov renewal theorem”, *Stocastic Processes and their Applications,* 50: 37–56.

Chang, C. S. (1993), “Stability, queue length and delay of deterministic stochastic queueing networks”, IBM Yorktown Heights, NY, T. J. Watson Research Center.

Chang, C. S., Heidelberger, P., Juneja, S., and Shahabuddin, P. (1992), “Effective bandwidth and fast simulation of ATM intree networks”, Yorktown Heights, NYIBM T. J. Watson Research Center.

Cohen, J. W. (1982), “The Single Server Queue”, North-Holland Publishing Company, Amsterdam.

Doshi, B. (1986), “Queueing systems with vacations-a survey”, *Queueing Systems,* 1: 29–66.

Falkenberg, E. (1994), “On the asymptotic behavior of the stationary distribution of Markov chains of M/G/1 type”, *Stochastic Models,* 10: 75–97.

Foley, R. D. and McDonald, D. R. (2005), “Large deviations of a modified Jackson network: stability and rough asymptotics”, *Annals of Applied Probability,* 15(1B): 519–541.

Glynn, P. W. and Whitt, W. (1994), “Logarithmic asymptotics for steady state tail probabilities in a single server queue”, *Advances in Applied Probability,* 31(A): 131–156.

Grassmann, W. K. and Heyman, D. P. (1990), “Equilibrium distribution of block-structured Markov chains with repeating rows”, *Journal of Applied Probability,* 27: 557–576.

Kimura, T., Daokoku, K., Masuyama, H., and Takahashi, Y. (2010), “Light-tailed asymptotics of the stationary tail probability vectors of Markov chains of the $M/G/1$-type”, *Stocastic Models,* 26(4): 505–548.

Kimura, T., Masuyama, H., and Takahashi, Y. (2012), “Heavy-traffic asymptotics for stationary GI/G/1-type Markov chains”, *Operations Research Letters,* 40(3): 185–189.

Kimura, T., Masuyama, H., and Takahashi, Y. (2013), “Subexponential Asymptotics of the Stationary Distributions of GI/G/1-Type Markov Chains”, *Stocastic Models,* 29(2): 190–239.

Li, Q and Zhao, Y. Q. (2005a), “Heavy-tailed asymptotics of stationary probability vectors of GI/G/1-Type”, *Advances in Applied Probability,* 37: 482–509.

Li, Q. and Zhao, Y. Q. (2005b), “Light-tailed asymptotics of stationary probability vectors of Markov chains of GI/G/1-Type”, *Advances in Applied Probability,* 37: 1075–1093.

Lucantoni, D. M. (1991), “New results on the single-server queue a batch arrival process”, *Stocastic Models,* 7(1): 1–46.

Lucantoni, D. M., Meier-Hellstern, K. S., and Neuts, M. F. (1990), “A single-server queue with server vacations and a class of non-renewal arrival processes”, *Advances in Applied Probability,* 22(3): 676–705.

Markushevich, A. I. (1965), “Theory of functions of a Complex variable”, Volumn IPrentice-Hall, Inc.

Masuyama, H. (2011), “Subexponential asymptotics of the stationary distributions of M/G/1-type Markov chains”, *European Journal of Operational Research,* 213(3): 509–516.

Miyazawa, M. (2009), “Tail decay rates in doubly QBD processes”, *Mathematics of Operations Research,* 34(3): 547–575.

Møller, J. R. (2001), “Tail asymptotics for M/G/1 type queueing processes with light-tailed increments”, *Operations Research Letters,* 28: 181–185.

Nakagawa, K. (2004), “On the exponential decay rate of the tail of a discrete probability distribution”, *Stocastic Models,* 20(1): 31–42.

Neuts, M. F. (1981), “Matrix-geometric solutions in stochastic models: an algorithmic approach”, BaltimoreThe Johns Hopkins University Press.

Neuts, M. F. (1989), “Structured Stochastic Matrices of M/G/1 Type and Their Applications”, New YorkMarcel Dekker.

Rudin, W. (1974), “Real and Complex Analysis”, McGraw-Hill, Inc.

Seneta, E. (1981), “Non-negative Matrices and Markov Chains”, New YorkSpringer-Verlag.

Tian, N. and Zhang, Z. (2006), “Vacation Queueing Models: Theory and Applications”, New YorkSpringer-Verlag.

Tai, Y. and Zhao, Y. Q. (2010), Exact asymptotics for the $GI/G/1$-type Markov chains, submitted.

Takine, T. (2004), “Geometric and subexponential asymptotics of Markov chains of $M/G/1$-type”, *Mathematics of Operations Research,* 29: 624–648.

Whitt, W. (1993), “Tail probabilities with statistical multiplexing and effective bandwidths for multi-class queues”, *Telecommunication Systems,* 2: 71–107.

Zhao, Y. Q., Li, Q., and Braun, W. J. (1998), “Infinite block-structured transition matrices and their properties”, *Advances in Applied Probability,* 30: 365–384.

Zhao, Y. Q. (2000), Latouche, B. and Taylor, P.“Censoring technique in studying block-structured Markov chains”, Advances in Algorithmic Methods for Stochastic Models, Notable Publications, 417–433.

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